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A Mathematical Development of Minimal Surface Theory: From Soap Films to Black Holes

Timothy Pitts

A MATHEMATICAL DEVELOPMENT OF MINIMAL SURFACE THEORY: FROM SOAP FILMS TO BLACK HOLES

A THESIS SUBMITTED TO THE

HONORS COLLEGE

IN PARTIAL FULFILLMENT OF

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BY

TIMOTHY PITTS

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ABSTRACT

Minimal surfaces are a special subset of surfaces that have gone through a long and extensive development and have also led to many fruitful findings in mathematics. Several periods that are key to the progression of the theory are coined as Golden Ages for the field's development. Here, a historical and mathematical development of minimal surface theory is presented that spans from its inception in the late 18th century to the present day. Along with the development, there is an emphasis on showing connections of minimal surfaces to various natural phenomena that occur such as soap films, black holes, biological systems, etc. Lastly, it is discussed briefly where the field is currently and where its future lies beyond.

ACKNOWLEDGEMENTS

I would like to thank the help and support from Dr. Carlos Almada for showing me the wonder of minimal surfaces and their mathematics.

I would also like to thank my family and friends for this study would not have been possible without the support and encouragement from them all.

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May 10, 2020

1 Introduction

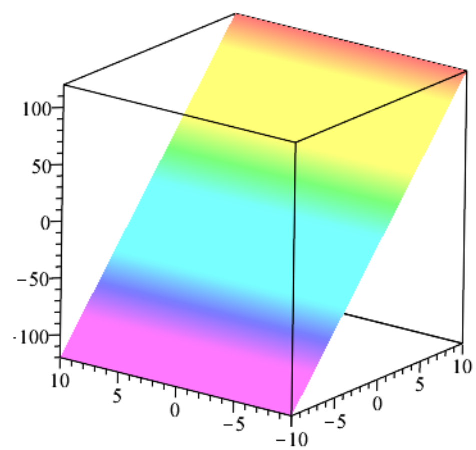
Minimal surface theory is a sub-branch of mathematics that has been in development for over two centuries. Through its inception, it has incorporated many broader branches of mathematics such as the calculus of variations, complex analysis, differential geometry, and mathematical physics. Minimal surfaces are a special subset of surfaces that can be described using several different but equivalent definitions, and they have their roots dating back to the late 17th century with the development of the calculus of variations by Joseph Louis Lagrange and Leonhard Euler. Since the development of the field, minimal surfaces have proven to be applicable to many disciplines such as materials engineering, architecture, physics, computer graphics, and biology as well as many others. More importantly, through the analysis of minimal surfaces, mathematicians have been able to formulate more precise notions of what is meant by concepts such as shape, curvature, and spatial relations. William H. Meeks III and Joaquin Perez describe their perspective on the formulation of minimal surface theory as progressing through several *Golden Ages* of development spanning from the early 19th century to the end of the 20th century, and they even claim that such a *Golden Age* is being witnessed currently as of the 1980s [23]. They coined the term *Classical Minimal Surface Theory* which primarily discusses minimal surfaces that are connected, orientable, complete, and embedded in $\mathbb{R}^3$ with a finite genus. In contrast, the *Modern Theory* explores surfaces in higher dimensional manifolds of a more complex nature. This thesis primarily addresses Meeks and Perez's perspective of the development of minimal surface theory while constructing a unique interpretation of the mathematical and historical progression of the theory. Throughout the development, there is a discussion of minimal surface applications in other disciplines and where research can progress in the field.

2 Overview of Minimal Surface Theory

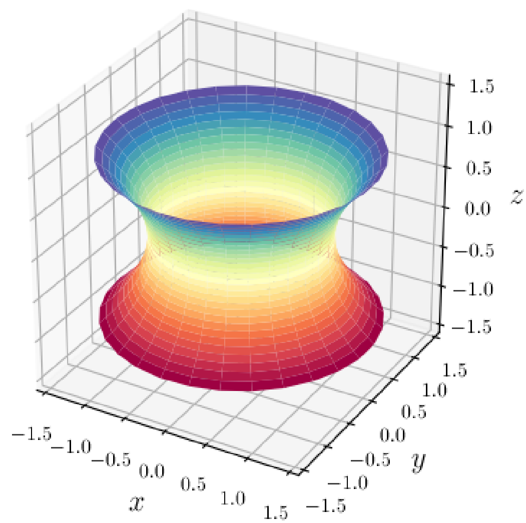
The theory of minimal surfaces can be further divided into two separate developments: classical minimal surface theory and modern minimal surface theory. Classical theory refers to the study of surfaces that are connected, orientable, complete, embedded minimal surfaces in $\mathbb{R}^3$ with a finite genus. Classical surfaces do not deal with singularities, where points either blow up to infinity or do not exist, and they also do not address surfaces that self intersect. There are however a few surfaces that will be seen later that do indeed allow for self intersections. In Meeks and Perez's brief survey of classical minimal surface theory, they mostly referenced work that built up to their solution of the problem that the plane, the helicoid, the catenoid, and the one-parameter family $\{R_t\}_{t\in(0,1)}$ of Riemann minimal examples are the only complete, properly embedded, minimal planar domains in $\mathbb{R}^3$. There will not be much emphasis on their particular problem above, but the theory in general will be explored. The first minimal surfaces known at the beginning of the theory were the plane, catenoid, and helicoid. The next section provides a visualization of the first three minimal surfaces known before the work of Heinrich Scherk in the 1800s.

2.1 Classical Surface Examples

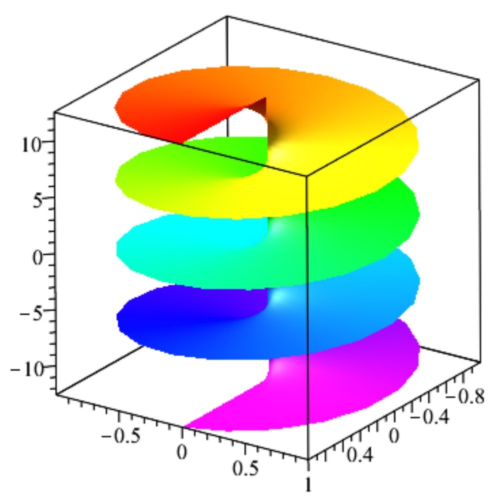
- **The Plane:** This surface resembles a flat sheet of paper. An example is the ordinary coordinate plane.



- **The Catenoid:** The only minimal surface of revolution. It is formed by rotating the curve known as the catenary about a central axis.



- **The Helicoid:** The only ruled minimal surface. A Ruled surface is one which can be constructed entirely out of straight lines.



As stated before, Meeks and Perez noted that classical minimal surface theory has gone through several *Golden Ages* of development, and they specify three main periods. The first period is roughly between the years 1850-1890. This period is highlighted with work from Enneper, Weierstrass, Scherk, Riemann, and Gauss and is primarily motivated by the formulation of complex variables and analysis. Also at this time, the Belgian physicist Joseph Plateau experimented with soap films to study their physical nature and their connection to mathematical surfaces. Then, around 1930-1950, there is another expansion of the theory. According to Robert Osserman, this period sought to expand on minimal surface problems related to partial differential equations and Joseph Plateau's problem for soap films [19]. The third and most recent period is argued to have started in the 1980s, and is when *Modern Minimal Surface Theory* begins to develop with use of mathematical fields such as geometric measure theory, manifolds and submanifolds, and non-parametric minimal surfaces. The three periods are used as a foundation for the literature related to the development of minimal surface theory. The review of the literature emphasizes each work's contribution to the overall significance to the mathematics of physical models, like soap films, and why the results are crucial for applications in other research areas.

2.2 The Calculus of Variations: 1690-1780

Before the first *Golden Age* in the 1800s, several Bernoulli relatives, Euler, and Lagrange developed an essential field called the calculus of variations. This new calculus, which developed shortly after ordinary calculus, helped pave the way for understanding the nature of shape for curves and surfaces. The study of variational calculus reveals how nature optimizes itself in a geometrical sense, with curves and surfaces, as well as in a physical sense with energy and motion.

To demonstrate how the calculus of variations quantifies the optimization of nature, consider the following problem. The first problem that fueled the creation of variational calculus was known as the Brachistochrone Problem. The problem asks to find a curve, that is only under the influence of gravity, and allows an object to travel from the highest point to the lowest point in the least amount of time. The answer was found to be part of a cycloid, and it was solved independently by several notable figures. For solutions, refer to [20]. A cycloid is a path traced out by a point on the edge of a circle as the circle is rolling with constant speed. It may seem arbitrary how a the cycloidal path of a rolling circle answers the Brachistochrone problem, but there is a simple way to connect the ideas.

The connection is Snell's Law. Discovered in 1621 by Willebrord Snell [6], the law states that as light passes through a boundary from one medium to another it is refracted according to the refractive indices of each medium. The relationship is described as

$$\frac{sin(\theta_1)}{sin(\theta_2)} = \frac{v_1}{v_2} \quad .$$

Where θ_1 and θ_2 are the angles of incidence to the perpendicular line drawn at the point of contact of the boundary, and v_1 and v_2 are the speeds that light travels through the different substance mediums. Johann Bernoulli's solution in 1696 of the Brachistochrone problem used Snell's law by taking several refractive prisms and stacking them in layers [20]. After stacking several against one other, the light beam kept refracting through each prism causing the path of the light to curve. After enough prisms, the beam's path began to approach what is now known as the Brachistochrone. Therefore, nature displays paths that are solutions to mathematical problems according to its own laws. Most importantly, the essential idea of the calculus of variations is to formulate how certain quantities are demanded to be optimized by nature. In the Brachistochrone problem, the desire is to get from one point to another in the shortest amount of time, thus time needs to be minimized. Overall, the calculus of variations attempts to find certain curves, surfaces, or paths that maximize or minimize some functional using calculus operations; namely through the use of integrals. The maximization and minimization of these integrals is more commonly known as extremizing an integral, and the point at which a functional is extremized it is called stationary. The Euler-Lagrange equations were formulated based on this idea of optimization to assist in finding solutions that would extremize a functional when it is stationary. This is almost similar to finding maximums and minimums of functions using a derivative of a function and setting it equal to zero. For surfaces, the optimization of area functionals that are desired to be extremized lead to a few definitions for a minimal surface. According to Meeks and Perez [23]:

Definition 2.2.1: A surface $M \subset \mathbb{R}^3$, is minimal if and only if it is a critical point of the area functional for all compactly supported variations.

Definition 2.2.2: A surface $M \subset \mathbb{R}^3$, is minimal if and only if every point $p \subset M$ has a neighborhood with least-area relative to its boundary.

Both of these definitions for a minimal surface are equivalent and are derived from the work accomplished by Euler and Lagrange. Note in **Definition 2.2.2**, a surface is minimal if it locally minimizes its area and creates a least-area neighborhood for every point relative to the surface's boundary. Thus, global area minimization for a surface is not a necessary condition for minimal surfaces but nearly a result for some minimal surfaces. Euler and Lagrange derived the previous definitions by finding necessary conditions for minimal surfaces.

2.3 Conditions for Minimal Surfaces:

Euler-Lagrange Equations: Let J denote a functional for the surface area that is to be extremized. The surface area functional is $J = \int\int\Big(\sqrt{1 + f_u^2 + f_v^2} dudv$, and if it is to be extremized then the Euler-Lagrange equations must hold at some stationary point. The Euler-Lagrange equation for two independent variables is as follows:

$$\frac{\partial f}{\partial x} - \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x_t}\right) - \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x_s}\right) = 0$$

Also, if we take a mapping $\mathbf{x}(u, v) = (u, v, f(u, v))$, then J can be used further to derive a special partial differential equation (PDE). Here is the derivation from [10].

$$\begin{aligned}
0 &= \frac{\partial \sqrt{1 + f_u^2 + f_v^2}}{\partial f} - \frac{\partial}{\partial u}\left(\frac{\partial(\sqrt{1 + f_u^2 + f_v^2})}{\partial f_u}\right) - \frac{\partial}{\partial v}\left(\frac{\partial(\sqrt{1 + f_u^2 + f_v^2})}{\partial f_v}\right) \\
&= 0 - \frac{\partial}{\partial u}\left(\frac{f_u}{\sqrt{1 + f_u^2 + f_v^2}}\right) - \frac{\partial}{\partial v}\left(\frac{f_v}{\sqrt{1 + f_u^2 + f_v^2}}\right) \\
&= \frac{f_{uu}(1 + f_u^2 + f_v^2) - f_u(f_u f_{uu} + f_v f_{uv})}{(1 + f_u^2 + f_v^2)^{3/2}} \\
&\quad + \frac{f_{vv}(1 + f_u^2 + f_v^2) - f_v(f_u f_{uv} + f_v f_{vv})}{(1 + f_u^2 + f_v^2)^{3/2}} \\
&= \frac{f_{uu}(1 + f_v^2) - 2 f_u f_v f_{uv} + f_{vv}(1 + f_u^2)}{(1 + f_u^2 + f_v^2)^{3/2}} \quad .
\end{aligned}$$

Since the left hand side of the equation is 0, then this leaves the numerator as equalling to 0 to satisfy this condition. Hence, The area functional reduces down to a single equation in terms of f according to the formula

$$f_{uu}(1 + f_v^2) - 2 f_u f_v f_{uv} + f_{vv}(1 + f_u^2) = 0 \quad .$$

The equation is commonly called the minimal surface equation, and its solutions are various f functions that make the equation true. The solutions to this equations are not easy to calculate, but it is a necessary condition for the surface itself if it is minimal. Thus, checking if a surface satisfies this condition can further verify if the surface is indeed minimal. The equation above indicates that this must hold true for minimal surfaces at every point on a surface. Another definition for a minimal surface can be formed from the necessary condition.

Definition: A surface $M \subset \mathbb{R}^3$, is minimal if and only if its mean curvature vanishes identically [11].

For further information on mean curvature, see section 3.2. The partial differential equation was found to be equivalent to the vanishing of mean curvature by Meusnier in 1776. The surfaces noted before, namely the plane, catenoid, and helicoid all satisfy this condition, and so do minimal surfaces found in the later centuries. With the results by Euler and Lagrange, two other mathematicians during this period contributed a very useful tool for minimal surface theory.

Gaspard Monge and Adrien-Marie Legendre, derived the Monge-Legendre representations for a minimal surface based on the use of complex analysis. This representation uses the following formulas [11]:

$$x = \phi'(a) + \psi'(b)$$
$$y = \phi(a) - a\phi'(a) + \psi(b) - b\psi'(b)$$
$$z = \int \left(\sqrt{-1 - a^2}\right) \phi''(a)da + \int \left(\sqrt{-1 - b^2}\right) \psi''(b)db.$$

These formulas involved the use of two complex functions, ϕ and ψ, that are special functions in complex analysis. Their special property is that they are analytic functions of a and b. The reason why these functions are special is discussed later with the Weierstrass-Enneper representations. At the time, these representation equations were virtually useless because complex analysis was not well understood in a real geometric sense or in general because the field was very new and still being theorized. Though the representations were not able to be used efficiently, they allow minimal surfaces to be represented with the use of complex functions thus connecting the two fields to each other. The connection of these two fields is primarily what drives the first age of minimal surface theory.

3 The 1st *Golden Age*: 1850-1890

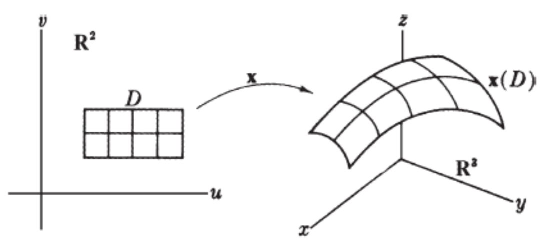
The first expansion of minimal surface theory came about in the study of differential geometry and complex variable analysis. The most notable advances were the formulation of geometry of curves and surfaces, study of soap films, and Enneper-Weierstrass representations for minimal surfaces.

The two fields that influenced much of minimal surface theory during this time period were Differential geometry and Complex Analysis. Differential geometry explores the quantification of curvature for surfaces in traditional Euclidean spaces as well as non-traditional geometries such as hyperbolic spaces. Here, in differential geometry, calculus is used to formulate spatial relations for surfaces and shapes. Complex analysis on the other provides a connection to minimal surfaces through its use of analytic and harmonic functions. Complex analysis is also constructed further during the 1800s allowing for notable contributions to minimal surface theory. It is necessary to discuss differential geometry and complex analysis in further detail to show how they contribute to the theory of minimal surfaces.

3.1 Surfaces in $\mathbb{R}^3$

Differential geometry describes the spatial mathematics of minimal surfaces, so mathematicians developed ways to describe or represent surfaces in a coordinate space. Many surfaces and curves are complex and cannot be described easily or understood with the traditional coordinate plane mappings or representations such as $y = f(x)$ and $z = f(x, y)$. Instead, curves and surfaces will be need to be re-written as parametrizations. A surface in $\mathbb{R}^3$ is, to begin with, a subset of $\mathbb{R}^3$, that is, a certain collection of points of $\mathbb{R}^3$. Of course, not all subsets are surfaces, but we must certainly require that a surfaces must be smooth and two-dimensional. These requirements will be expressed in mathematical terms in the following paragraphs.

Let $\mathbf{x} : D \to \mathbb{R}^3$ be a differentiable mapping of an open set D of $\mathbb{R}^2$ into $\mathbb{R}^3$. The domain D will usually be an open disk or an open rectangle. If $(u, v) \in D$, then $\mathbf{x}(u, v) = (x^1(u, v), x^2(u, v), x^3(u, v))$. This mapping is called a *parametrization* or a *coordinate patch* and the $x^i(u, v)$ are called component functions of $\mathbf{x}$. Under certain conditions we describe below, the image $\mathbf{x}(D)$ of a coordinate patch $\mathbf{x}$, that is the set of all values of $\mathbf{x}$, is a smooth two-dimensional subset of $\mathbb{R}^3$. The following figure illustrates the idea.



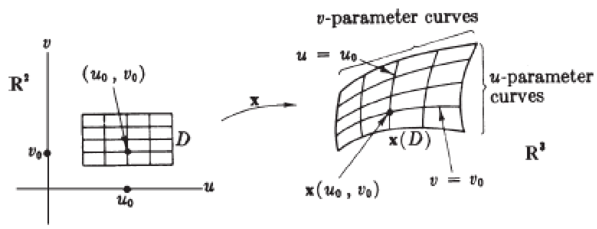
Let $\mathbf{x} : D \to \mathbb{R}^3$ be a coordinate patch. Holding u or v constant in the function $(u, v) \to \mathbf{x}(u, v)$ produces curves. Explicitly, for each point (u_0, v_0) in D the curve

$$u \to \mathbf{x}(u, v_0)$$

is called the *u-parameter curve*, $v = v_0$, of $\mathbf{x}$; and the curve

$$v \to \mathbf{x}(u_0, v)$$

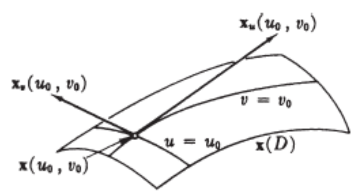
is called the *v-parameter curve*, $u = u_0$. The figure shows what these curves look like in general.



Thus, the image $\mathbf{x}(D)$ is covered by these two families of curves, which are the images under $\mathbf{x}$ of the horizontal and vertical lines in D, and one curve from each family goes through each point of $\mathbf{x}(D)$. The tangent vectors for the u-parameter and v-parameter curves are given by differentiating the component functions of $\mathbf{x}(u, v)$ with respect to u and v respectively. At the point $(u_0, v_0) \in D$, we have

$$\mathbf{x}_u(u_0, v_0) = \left(\frac{\partial x^1}{\partial u}, \frac{\partial x^2}{\partial u}, \frac{\partial x^3}{\partial u}\right)\Bigg(_{(u_0,v_0)} \quad \mathbf{x}_v(u_0, v_0) = \left(\frac{\partial x^1}{\partial v}, \frac{\partial x^2}{\partial v}, \frac{\partial x^3}{\partial v}\right)\Bigg(_{(u_0.v_0)}$$

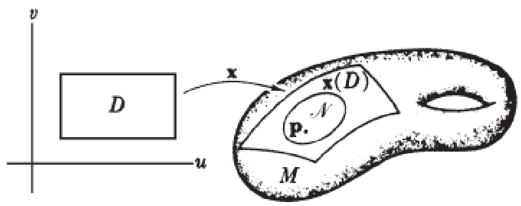
These tangent vectors or velocity vectors of the parameter curves are illustrated in the following figure.



Of course, to obtain true coordinates on a surface, we need two properties: first, $\mathbf{x}(u, v)$ must be one-to-one (although we can relax this condition slightly to allow for certain self-intersections of a surface); secondly, $\mathbf{x}(u, v)$ must never have $\mathbf{x}_u$ and $\mathbf{x}_v$ in the same direction because this destroys 2-dimensionality. That is, we need these velocity vectors to be linearly independent. When these two conditions are satisfied, we may say that the coordinate patch $\mathbf{x}(u, v)$ is regular. In order to avoid certain technical difficulties, we must use *proper* patches, those for which the inverse of $\mathbf{x} : \mathbf{x}(D) \to D$ is continuous (that is, has continuous coordinate functions). If we think of D as a thin sheet of rubber, then $\mathbf{x}(D)$ is created by bending and stretching D in a not too violent fashion.

To construct a suitable definition of a surface we start from the rough idea that any small enough region in a surface M resembles a region in the plane $\mathbb{R}^2$. The discussion above shows that this can be stated somewhat more precisely as, near each of its points, M can be expressed as the image of a proper patch. (When the image of the patch $\mathbf{x}$ is contained in M, we say that $\mathbf{x}$ is a patch in M). To get the final form of the definition, it remains only to define a neighborhood N of $\mathbf{p}$ in M to consist of all points of M whose Euclidean distance from $\mathbf{p}$ is less than some number $\epsilon > 0$.

Definition: A surface in $\mathbb{R}^3$ is a subset M of $\mathbb{R}^3$ such that for each point $\mathbf{p}$ of M there exists a proper patch in M whose image contains a neighborhood of $\mathbf{p}$ in M.



Example: Let us show that the unit sphere

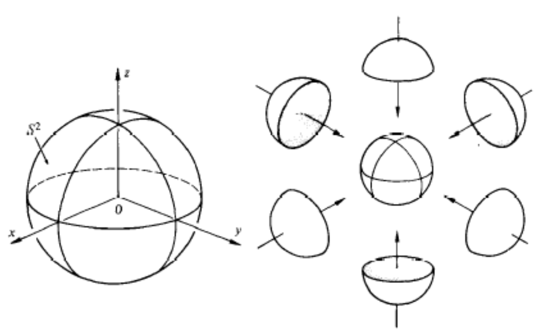
$$S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$$

is a surface.

We will construct the unit sphere with six proper patches without going into details. The patches, $\mathbf{x}_{+1}, \mathbf{x}_{-1}, \mathbf{x}_{+2}, \mathbf{x}_{-2}, \mathbf{x}_{+3}, \mathbf{x}_{-3}$ are given by

$$\mathbf{x}_{\pm 1}(u, v) = (u, v, \pm\sqrt{1 - u^2 - v^2})$$
$$\mathbf{x}_{\pm 2}(u, v) = (u, \pm\sqrt{1 - u^2 - v^2}, v)$$
$$\mathbf{x}_{\pm 3}(u, v) = (\pm\sqrt{1 - u^2 - v^2}, u, v)$$

Where $(u, v) \in D = \{(u, v) \in \mathbb{R}^2 | u^2 + v^2 < 1\}$. On this open region D, the function $\sqrt{1 - u^2 - v^2}$ is continuous and has partial derivatives. It is clear that each patch is one and that their inverses are the projections onto the xy-plane, xz-plane, and yz-plane, respectively. For instance, $\mathbf{x}_{\pm 1}^{-1} : S^2 \to D$ with $\mathbf{x}_{\pm 1}^{-1}(x, y, z) = (x, y)$. This inverse is continuous because of the restriction to S^2 of the continuous projection $\pi(x, y, z) = (x, y)$ from $\mathbb{R}^3$ onto $\mathbb{R}^2$. A similar argument applies in the other cases. The figure suggests that the unit sphere is obtained by patching together the six images of the proper patches defined above.



3.1.1 Tangent Vectors and Tangent Space

Let M be a surface and let I denote some interval in $\mathbb{R}$. If $\alpha : I \to \mathbf{x}(D) \subseteq M$ is a smooth curve in $\mathbb{R}^3$ which is contained in the image of a parametrization $\mathbf{x}$ on M, then there exists unique smooth functions $u(t), v(t) : I \to \mathbb{R}$ such that $\alpha(t) = \mathbf{x}(u(t), v(t))$.

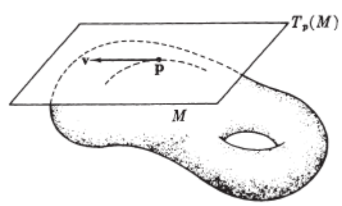
To see this let $(u(t), v(t)) = \mathbf{x}^{-1}\alpha(t)$, hence, $\alpha(t) = \mathbf{x}(\mathbf{x}^{-1}\alpha(t)) = \mathbf{x}(u(t), v(t))$.

These functions $u(t)$, $v(t)$ are called the coordinate functions of the curve α with respect to the patch $\mathbf{x}$.

It is intuitively clear what it means for a vector to be tangent to a surface M in $\mathbb{R}^3$. A formal definition can be based on the idea that a curve in M must have all its velocity vectors tangent to M.

Definition: Let $\mathbf{p}$ be a point on a surface M in $\mathbb{R}^3$. A vector $\mathbf{v}$ in $\mathbb{R}^3$ at the point $\mathbf{p}$ is *tangent* to M at $\mathbf{p}$ provided $\mathbf{v}$ is a velocity vector of some curve in M.

The set of all tangent vectors to M at $\mathbf{p}$ is called the tangent plane of M at $\mathbf{p}$ and is denoted by $T_p(M)$. The following result shows, in particular, that at each point $\mathbf{p}$ of M the tangent plane $T_p(M)$ is actually a 2-dimensional vector subspace of the tangent space $T_p(\mathbb{R}^3)$.



Let $\mathbf{p}$ be a point of a surface M in $\mathbb{R}^3$, and let $\mathbf{x}$ be a patch in M such that $\mathbf{x}(u_0, v_0) = \mathbf{p}$. A tangent vector $\mathbf{v}$ to $\mathbb{R}^3$ at $\mathbf{p}$ is tangent to M if and only if $\mathbf{v}$ can be written as a linear combination of $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$. Since partial velocities are always linearly independent, we deduce that they provided a *basis* for the tangent plane of M at each point of $\mathbf{x}(D)$.

The proof of this is as follows. First, note that the parameter curves of $\mathbf{x}$ are curves in M, so $\mathbf{x}_u$ and $\mathbf{x}_v$ are always tangent to M at $\mathbf{p}$. Now suppose that $\mathbf{v}$ is tangent to M at $\mathbf{p}$; thus, there is a curve α in M such that $\alpha(0) = \mathbf{p}$ and $\alpha'(0) = \mathbf{v}$. Since α may be written as $\alpha(t) = \mathbf{x}(u(t), v(t))$, by the chain rule we have

$$\alpha'(t) = \mathbf{x}_u(u(t), v(t))u'(t) + \mathbf{x}_v(u(t), v(t))v'(t) \quad .$$

Since $\alpha(0) = \mathbf{p} = \mathbf{x}(u_0, v_0)$ we have $(u(0), v(0)) = (u_0, v_0)$. Hence at $t = 0$ we have

$$\mathbf{v} = \alpha'(0) = \mathbf{x}_u(u_0, v_0)u'(0) + \mathbf{x}_v(u_0, v_0)v'(0) \quad .$$

Conversely, suppose that a tangent vector $\mathbf{v}$ to $\mathbb{R}^3$ can be written as

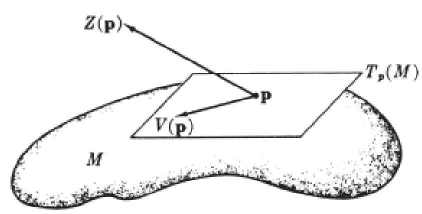
$$\mathbf{v} = c_1\mathbf{x}_u(u_0, v_0) + c_2\mathbf{x}_v(u_0, v_0)$$

By computations as above, $\mathbf{v}$ is the velocity vector at $t = 0$ of the curve $\alpha(t) = \mathbf{x}(u_0 + tc_1, v_0 + tc_2)$.

A reasonable deduction, based on the general properties of derivatives, is that the tangent plane $T_P(M)$ is the linear approximation of the surface M near $\mathbf{p}$.

Definition: A Euclidean vector field Z on a surface M in $\mathbb{R}^3$ is a function that assigns to each point $\mathbf{p}$ of M a vector $Z(\mathbf{p})$ in $\mathbb{R}^3$ at $\mathbf{p}$.

A Euclidean vector field V for which each vector $V(\mathbf{p})$ is tangent to M at $\mathbf{p}$ is called a *tangent vector field* on M (see figure below). Frequently these vector fields are defined, not on all of M, but only on some region in M. As usual, we always assume differentiability. A Euclidean vector $\mathbf{z}$ at a point $\mathbf{p}$ of M is *normal* to M if it is orthogonal to the tangent plane $T_P(M)$, that is, to every tangent vector to M at $\mathbf{p}$, and a Euclidean vector field Z on M is a *normal vector field* on M provided each vector $Z(\mathbf{p})$ is normal to M.



3.1.2 The Shape Operator of $M \subset \mathbb{R}^3$

Suppose that Z is a Euclidean vector field on a surface M in $\mathbb{R}^3$. Let α be a curve in M with $\alpha(0) = \mathbf{p}$ and initial velocity $\alpha'(0) = \mathbf{v}$. Then, we define the covariant derivative of the vector field Z in the direction of $\mathbf{v}$ at the point $\mathbf{p}$, denoted by $\nabla_{\mathbf{v}}Z$, by

$$\nabla_{\mathbf{v}}Z = \tfrac{d}{dt}Z(\alpha(t))|_{t=0} \quad .$$

That is, $\nabla_{\mathbf{v}}Z$ is the rate of change of Z in the $\mathbf{v}$ direction at $\mathbf{p}$. If $Z = \sum_i z_i E_i$, where $\{E_1, E_2, E_3\}$ is the natural frame field of $\mathbb{R}^3$, then

$$Z(\alpha(t)) = \sum_i z_i(\alpha(t))E_i$$

and

$$\nabla_{\mathbf{v}}Z = \tfrac{d}{dt}Z(\alpha(t))|_{t=0} = \sum_i (z_i \circ \alpha)'(0)E_i \quad .$$

We now consider a specific vector field on M, namely, a *unit normal vector field* U on M. If $\mathbf{x} : D \to \mathbb{R}^3$ is a coordinate patch, then we can always construct such a field U on $\mathbf{x}(D)$ by letting

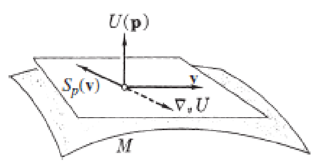
$$U = \frac{\mathbf{x}_u \times \mathbf{x}_v}{||\mathbf{x}_u \times \mathbf{x}_v||} \quad .$$

We are now in a position to find the mathematical measurement of the shape of a surface in $\mathbb{R}^3$.

Definition: If $\mathbf{p}$ is a point on M, then for each tangent vector $\mathbf{v}$ to M at $\mathbf{p}$, let

$$S_{\mathbf{p}}(\mathbf{v}) = -\nabla_{\mathbf{v}} U \quad ,$$

where U is a unit normal vector field on a neighborhood of $\mathbf{p}$ in M. $S_{\mathbf{p}}$ is called the *shape operator* of M at $\mathbf{p}$ derived from U. The figure illustrates the concept.



The tangent plane of M at any point $\mathbf{q}$ consists of all Euclidean vectors orthogonal to $U(\mathbf{q})$. Thus, the rate of change $\nabla_{\mathbf{v}} U$ of U in the $\mathbf{v}$ direction tells how the tangent planes of M are varying in the $\mathbf{v}$ direction, and this gives an infinitesimal description of the way M itself is curving in $\mathbb{R}^3$.

An important observation about the shape operator is the following. For each point $\mathbf{p}$ of the surface M, the shape operator is a linear operator

$$S_{\mathbf{p}} : T_{\mathbf{p}}(M) \to T_{\mathbf{p}}(M)$$

on the tangent plane of M at $\mathbf{p}$. Moreover, the shape operator is a symmetric operator with respect to the usual dot product of vectors in $\mathbb{R}^3$. That is,

$$S_{\mathbf{p}}(\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot S_{\mathbf{p}}(\mathbf{w})$$

for any pair of tangent vectors $\mathbf{v}$ and $\mathbf{w}$ to M at $\mathbf{p}$.

3.2 Curvature

Using the parametrizations of curves and surfaces, there is now the concept of defining what exactly curvature is and how it can be described mathematically. Most people have an intuitive notion of what curvature is, but the essential problem is formulating it. There are several types of curvature that are used within the study of minimal surfaces or surfaces in general.

3.2.1 The Normal Curvature

Throughout this section we will work in a region of a surface M that has been *oriented* by the choice of a unit normal vector field U, and we use the shape operator S derived from U. The shape of a surface in $\mathbb{R}^3$ influences the shape of the curves in M.

Lemma If α is a curve in M, then

$$\alpha'' \cdot U = S(\alpha') \cdot \alpha'$$

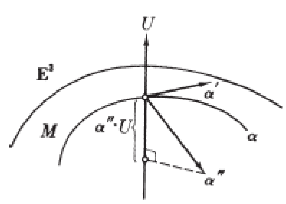
Proof Since α is in M, its velocity α' is always tangent to M. Thus, $\alpha' \cdot U(\alpha) = 0$. Differentiating this with respect to t, we get

$$\alpha'' \cdot U(\alpha) + \alpha \cdot (U(\alpha))' = 0$$

But $S(\alpha') = -(U(\alpha))'$, hence the result.

Geometric interpretation: at each point, $\alpha'' \cdot U$ is the component of the acceleration α'' normal to the surface M (see figure below). The lemma shows that this component depends only on the velocity α' and the shape operator of M. Thus, all curves in M with a given velocity vector $\mathbf{v}$ at point $\mathbf{p}$ will have the same normal component of acceleration at $\mathbf{p}$, namely, $S_{\mathbf{p}}(\mathbf{v}) \cdot \mathbf{v}$. This is the component of acceleration that the bending of M in $\mathbb{R}^3$ forces them to have. Thus, if $\mathbf{v}$ is standardized by reducing it to a unit vector u, we get a measurement of the way M is bent in the $\mathbf{u}$ direction.

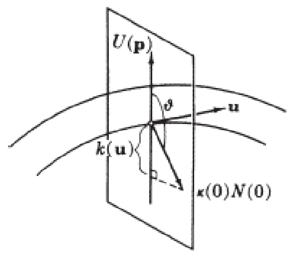
Definition: Let $\mathbf{u}$ be a unit vector tangent to M at a point $\mathbf{p}$. Then the number $k(\mathbf{u}) = S_{\mathbf{p}}(\mathbf{u}) \cdot \mathbf{u}$ is called the *normal curvature* of M in the $\mathbf{u}$ direction.



Given a unit tangent vector $\mathbf{u}$ to M at $\mathbf{p}$, let a be a unit-speed curve in M with initial velocity $\alpha'(0) = \mathbf{u}$. Using the Frenet apparatus of a, the preceding lemma gives

$$k(\mathbf{u}) = S_{\mathbf{p}}(\mathbf{u}) \cdot \mathbf{u} = \alpha''(0) \cdot U(\mathbf{p}) = k(0)N(0) \cdot U(\mathbf{p}) = k(0)cos(\theta)$$

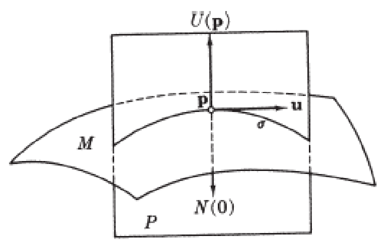
Thus, the normal curvature of M in the $\mathbf{u}$ direction is $k(0)cos(\theta)$, where $k(0)$ is the curvature of α at $\alpha(0) = \mathbf{p}$, and θ is the angle between the principal normal $N(0)$ and the surface normal $U(\mathbf{p})$, as in the figure.

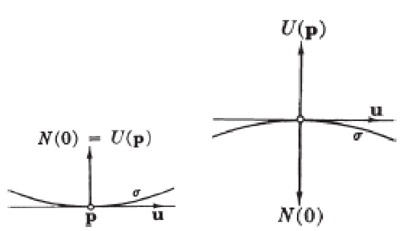


Given **u**, there is a natural way to choose the curve so that θ is 0 or π. In fact, if P is the plane determined by **u** and $U(\mathbf{p})$, then P cuts from M (near **p**) a curve σ called the *normal section* of M in the **u** direction. If we give σ unit-speed parametrization with $\sigma'(0) = \mathbf{u}$, then $N(0) = \pm U(\mathbf{p})$, since $\sigma''(0) = k(0)N(0)$ is orthogonal to $\sigma'(0) = \mathbf{u}$ and tangent to the plane P.

So, for a normal section in the **u** direction (see figure below)

$$k(\mathbf{u}) = k_\sigma(0)N(0) \cdot U(\mathbf{p}) = \pm k_\sigma(0) \quad .$$





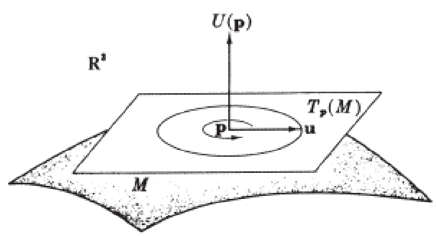
Thus, it is possible to make a reasonable estimate of the normal curvatures in various directions on a surface M by picturing what the corresponding normal sections would look like. We know that the principal normal N of a curve tells in which direction it is turning. Thus, the preceding discussion gives geometric meaning to the sign of the normal curvature $k(\mathbf{u})$ (relative to our fixed choice of U).

(1) If $k(\mathbf{u}) > 0$, then $N(0) = U(\mathbf{p})$, so the normal section σ is bending toward $U(\mathbf{p})$ at **p** (see figure above). Thus, in the **u** direction the surface M is bending toward $U(\mathbf{p})$.

(2) If $k(\mathbf{u}) < 0$, then $N(0) = -U(\mathbf{p})$, so the normal section σ is bending away from $U(\mathbf{p})$ at $\mathbf{p}$. Thus, in the $\mathbf{u}$ direction M is bending away from $U(\mathbf{p})$ (see figure above).

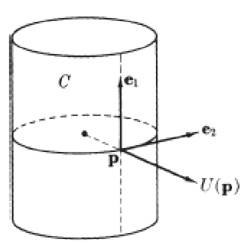
(3) If $k(\mathbf{u}) = 0$, then $k_\sigma(0) = 0$ and $N(0)$ is undefined. Here the normal section σ is not turning at $\sigma(0) = \mathbf{p}$. We cannot conclude that in the $\mathbf{u}$ direction M is not bending at all, since k might be zero only at $\sigma(0) = \mathbf{p}$. But we can conclude that its rate of bending is unusually small.

Let us now fix a point $\mathbf{p}$ of M and imagine that a unit tangent vector $\mathbf{u}$ at $\mathbf{p}$ revolves, sweeping out the unit circle in the tangent plane $T_p(M)$. From the corresponding normal sections, we get a moving picture of the way M is bending in every direction at $\mathbf{p}$ (see figure below).



Definition: Let $\mathbf{p}$ be a point of M. The maximum and minimum values of the normal curvature $k(\mathbf{u})$ of M at $\mathbf{p}$ are called the *principal curvatures* of M at $\mathbf{p}$, and are denoted by k_1 and k_2. The directions in which these extreme values occur are called *principal directions* of M at $\mathbf{p}$. Unit vectors in these directions are called *principal vectors* of M at $\mathbf{p}$.

Using the normal-section scheme discussed above, it is often fairly easy to pick out the directions of maximum and minimum bending. For example, if we use the outward normal U on a circular cylinder C as in the figure below, then the normal sections of C all bend away from U, so $k(\mathbf{u}) \leq 0$. Furthermore, it is reasonably clear that the maximum value $k_1 = 0$ occurs only in the direction $\mathbf{e}_1$ of a ruling; minimum value $k_2 < 0$ occurs only in the direction $\mathbf{e}_2$ tangent to a cross-section.



An interesting special case occurs at points $\mathbf{p}$ for which $k_1 = k_2$. The maximum and minimum normal curvature being equal, it follows that $k(\mathbf{u})$ is constant: M bends the same amount in all directions at $\mathbf{p}$ (so all directions are principal). In this case, the point $\mathbf{p}$ is called an umbilic point of M. For instance, every point on the sphere of radius $\mathbf{r}$ is an umbilic point with $k_1 = k_2 = -1/\mathbf{r}$.

We now state a very important result concerning the shape operator.

Theorem

(1) If $\mathbf{p}$ is an umbilic point of M, then the shape operator S at $\mathbf{p}$ is just scalar multiplication by $k = k_1 = k_2$.

(2) If $\mathbf{p}$ is a non-umbilic point, $k_1 \neq k_2$, then there are exactly two principal directions, and these are orthogonal. Furthermore, if $\mathbf{e}_1$ and $\mathbf{e}_2$ are principal vectors in these directions, then

$$S_{\mathbf{p}}(\mathbf{e}_1) = k_1\mathbf{e}_1 \text{ and } S_{\mathbf{p}}(\mathbf{e}_2) = k_2\mathbf{e}_2 \quad .$$

In short, the principal curvatures of M at $\mathbf{p}$ are the eigenvalues of S, and the principal vectors of M at $\mathbf{p}$ are the *eigenvectors* of S.

Through some translations and change of coordinates in $\mathbb{R}^3$, it is possible to show that the shape of M near a point $\mathbf{p}$ is approximately the same as that of the surface M' given by

$$z = \frac{1}{2}(k_1 x^2 + k_2 y^2) \quad .$$

Here, $\mathbf{p}$ is at the origin and the x and y axes are the principal directions at $\mathbf{p}$. M' is called the *quadratic approximation* of M near $\mathbf{p}$.

3.2.2 Gaussian Curvature

The preceding section found the geometrical meaning of the eigenvalues and eigenvectors of the shape operator. Now we examine the determinant and trace of S.

Definition: The *Gaussian curvature* of M is the real-valued function $K = det(S)$ on M. Explicitly, for each point $\mathbf{p}$ of M, the Gaussian curvature $K(\mathbf{p})$ of M at $\mathbf{p}$ is the determinant of the shape operator S of M at $\mathbf{p}$.

The *mean curvature* of M is the function $H = \frac{1}{2}trace(S)$. Gaussian and mean curvature are expressed in terms of principal curvature by

Lemma $K = k_1 k_2$ and $H = \frac{1}{2}(k_1 + k_2)$.

Proof The determinant (and trace) of a linear operator may be defined as the common value of the determinant (and trace) of all its matrices. If $\mathbf{e}_1$ and $\mathbf{e}_2$ are principal vectors at a point $\mathbf{p}$, then by a previous result, we have $S_{\mathbf{p}}(\mathbf{e}_1) = k_1(\mathbf{p})\mathbf{e}_1$ and $S_{\mathbf{p}}(\mathbf{e}_2) = k_2(\mathbf{p})\mathbf{e}_2$. Thus, the matrix of S at $\mathbf{p}$ with respect to $\mathbf{e}_1$, $\mathbf{e}_2$ is

$$\begin{bmatrix} k_1(\mathbf{p}) & 0 \\ 0 & k_2(\mathbf{p}) \end{bmatrix}.$$

This immediately gives the required result.

During the 1800s, Carl Friedrich Gauss formulated a special theorem called *Gauss' Theorema Egregium*. His theorem states that Gaussian curvature depends primarily on the nature of the surface itself and is independent from how a surface is embedded or placed in space [1]. This implies that Gaussian curvature is intrinsic to a surface rather than it being extrinsic. A great example to demonstrate this theorem is to compare a cylinder to a plane. For any given point on a cylinder, the principle curvature is some positive constant say k_1 and the other is 0 by the construction of a cylinder. Hence, the product of these curvatures is 0 meaning the Gaussian curvature is 0 for that surface. The plane also has 0 Gaussian curvature because of its shape. By Gauss' theorem a cylinder and plane are the same but just oriented differently in Euclidean space. This was a remarkable theorem for the field of differential geometry because intrinsic properties of surface can convey much information about various surfaces.

From the definitions of curvature we can define a minimal surface.

Definition: A surface is said to be a *minimal surface* if for every point on the surface the *mean curvature*, $H = 0$. This is also referred to as the vanishing of the *mean curvature*.

This definition is consistent with the minimal surface equation. To simplify the idea of the curvature definitions, think of a surface's curvature as being approximated by some circle or sphere that best approximates the curve or surface at a point, depending on if it is in $\mathbb{R}^2$ or $\mathbb{R}^3$. Typically then, the actual curvature for the point will be the reciprocal of the radius for the circle or sphere approximation. Sections 3.1 and 3.2 lay the mathematical foundation for the advancements of spatial geometry for minimal surface theory in the 1800s.

3.3 Soap Films and Plateau's Problem

Around 1870, The Belgian physicist, Joseph Plateau, experimented with soap films and soap bubbles to explain their physical nature. Soap films are created by a mixture of water and soap, and the addition of glycerin or corn syrup creates larger, more durable films and bubbles. Since water is a polar molecule, it has a molecular geometry that creates an unequal distribution of charge. Namely, the hydrogen atoms have a slight positive charge, and the oxygen atom has a slight negative charge. This unequal balance of charge causes water molecules to distribute an attractive force to other neighboring water molecules since the positive ends of hydrogen will attract the negatively charged ends of oxygen. This attraction causes a curvature to form on the surface, say a water droplet on a plant's surface for instance, because the molecules of water near the surface will feel a stronger attraction of force coming from inside the liquid rather than from the air molecules on the outside of the liquid. Hence, a curvature of the surface is formed. This property of pulling the surface of a liquid taut is called *surface tension*. See [4] for more details. The *surface tension* of a liquid can be altered by something called a surfactant. A surfactant usually lowers the *surface tension* of a liquid, thus soap is the surfactant in this solution. Soap is composed of molecules that have a hydrophilic and hydrophobic ends meaning water attracted and non-water attracted respectively. The hydrophilic ends remain in the soap-water solution, and the hydrophobic ends stick out of the solution.

In his experiments, Plateau was able to theorize several laws of soap films that were later proven mathematically. His laws are the following:

- The 120° Rule: Only three smooth surfaces of a soap film can meet along a line and the angle between any two of the three intersecting surfaces is 120°.
- 109° 28' Rule: Only four lines, each formed by the intersection of three surfaces, can meet at a point and the angle between any pair of adjacent lines is $arccos(-1/3) \approx 109°28'$.
- The 90° Rule: A soap film which is free to move along a surface meets the surface at right angles.

For more information regarding the laws and proofs of them refer to [18].

To mathematically discuss soap films and surface tension, mathematicians P.S. Laplace and Thomas Young derived an equation around 1800 that relates surface tension, the pressure difference on either side of the film, and the shape of the film itself [10]. This is the equation: $p = \sigma(\frac{1}{R_1} + \frac{1}{R_2})$, where p is the pressure difference on the sides of the film, σ is the surface tension of the substance, and $\frac{1}{R_1}$ and $\frac{1}{R_2}$ are the normal curvatures associated to any two perpendicular directions at a given point of the soap film surface. We can also write, based on the earlier definition of *mean curvature* in section 3.2, that $p = 2\sigma H$ because the normal curvatures $\frac{1}{R_1}$ and $\frac{1}{R_2}$ are the principal curvatures k_1 and k_2. Thus, the Laplace-Young equation allows soap films to be formulated purely with mathematics. It is important to note that this is one of the most fundamental equations for the mathematics of soap films. The consequence of this equation includes concluding that all soap films are physical models of minimal surfaces, and any soap film is a physical model of a local area minimizing surface as well as a surface of least area. This idea of a least area soap film model led to the problem known as Plateau's Problem.

Plateau's Problem asks if it is possible to find a surface M that is minimal for any given boundary C. The problem also concerns the existence and uniqueness of solutions for a given boundary. It is important to note that when the term minimal is applied to a surface it does not imply that the said surface, with whatever constraints, is a surface of least area. Being a minimal surface and being a surface of least area are quite different, but are often confused because they have some overlap when they are discussed. More specifically, for a surface to be minimal means that the surface locally minimizes its area, but it does not necessarily minimize the global area of a surface. This is why soap bubbles will not technically be called minimal surfaces even thought they are the shape that minimizes area for some arbitrary volume amount. For reference, remember the mantra that all surfaces of least area are minimal surfaces, but not all minimal surfaces are surfaces of least area. Weak solutions to Plateau's problem were formulated not long after its proposal, but the first global solution was not made until the 1930s by Jesse Douglas. At this time, soap films were the first real physical models of minimal surfaces. Along with the study of soap films, complex analysis allowed more surfaces that are minimal to be discovered.

3.4 Complex Analysis Connections

The subject of complex analysis connects to minimal surfaces in the use of what is known as holomorphic and harmonic functions. A function f is complex differentiable at a point z_0. Here z_0 is an element of the complex plane, and we say the function $f(z_0)$ is differentiable if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad .$$

Additionally, this limit must exist for all elements on the domain in order for it to be complex differentiable. It is then called a *holomorphic* function. A *holomorphic* function can be written as a function that is of a real part and an imaginary part. The real and imaginary parts for a holomorhpic function are each a harmonic function meaning that they satisfy the Cauchy-Riemann equations.

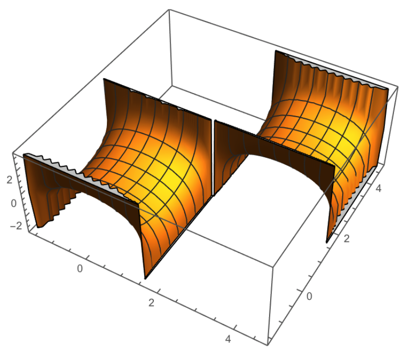
The **Weierstrass-Enneper Representation** are a set of equations that represent a connection between holomorphic functions and minimal surfaces. Here is the set of formula representations according to [24]:

If f is holomorphic, g is meromorphic, and fg^2 is holomorphic on a given set domain D, then a minimal surface is defined by the parametrization $\mathbf{x}(z, \bar{z}) = (x^1(z, \bar{z}), (x^2(z, \bar{z}), (x^3(z, \bar{z}))$, where

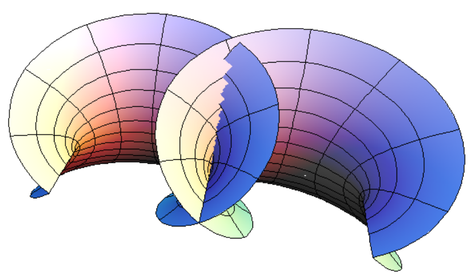
$$\begin{aligned} x^1(z, \bar{z}) &= \mathrm{Re} \int f(1 - g^2) dz, \\ x^2(z, \bar{z}) &= \mathrm{Re} \int i f(1 + g^2) dz, \\ x^3(z, \bar{z}) &= \mathrm{Re}\, 2 \int f g dz. \end{aligned}$$

From these formulas we can generate minimal surfaces by finding holomorphic functions. This was an enormous break through for minimal surfaces theory because it not only connected the surfaces to complex analysis, which at the time was just beginning to be created, but it also made it easier to generate new minimal surfaces. Thus, several new minimal surfaces were found during this period namely by Heinrich Scherk. Some examples follow.

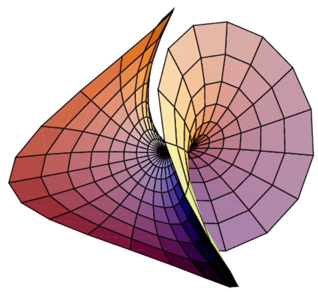
- **Scherk's** 1^{st} **Surface:**This surface was generated with the use of the Monge-Legendre representation. Scherk also found several other minimal surfaces during this time.



- **Catalan's Surface:** One of the self-intersecting surfaces that was found using the revolutionary Weierstrass-Enneper representation.



- **Enneper's Surface:** Of course Enneper used the representations noted earlier to find this minimal surface.



4 The 2nd *Golden Age*: 1930-1950

The largest achievements in the second age were the solution to Plateau's problem, the solution to Bernstein's Theorem, and the analysis of other partial differential equations. Plateau's Problem proved to be a very difficult problem in the field of minimal surface theory because it involved continuous boundaries of non-linear orientation and it was also a global theory problem. A boundary of continuity of non-linear orientation here means that the boundary is not necessarily constructed out of straight lines and instead can be of a circular shape. The problem is also a global problem because it concerns the behavior of least area surfaces which must globally minimize area. Recall that not all minimal surfaces are of least area. Those that are not of least area minimize area locally around each point, but not globally. An example that demonstrates how minimal surface do not always minimize surface area is seen in section 5.6 of [10]. For a much more in depth difference between least area surfaces and minimal surfaces see [17]. The property of a surface achieving global minimization of area is what concerns Plateau's Problem.

The solution of Plateau's problem came from two independent mathematicians at slightly different times during this century: Jesse Douglas [9] and Tibor Rado [21]. The first solution by Douglas was nearly a century after its proposal by Plateau which further demonstrates the complexity of the problem. One of the main advantages of the solution to the problem was the fact that it was a global solution which very few problems in minimal surface theory involve global solutions.

Historically, during the 20th century Albert Einstein formulated his Special and General Theory's of relativity. Minimal surfaces do not directly have much influence on his theory, but Einstein's theory is predicated on the use of

Riemannian Geometry and manifolds which thereby concern how nature conforms to certain geometrical objects or surfaces. His theory, in a sense, explains how space and time exist as a fabric where all matter resides. The mass of an object warps the fabric of space-time, and causes the fabric to curve much like a bowling ball resting on a trampoline's surface. That said, it has been shown earlier how nature tends to optimize geometry, so there must be some way that minimal surfaces exist in physical phenomena. Einstein's theory theorizes that if a mass is significant enough it could cause space-time to curve drastically creating a singularity more commonly known as a black hole. That said, since the early 1900s, Einstein's theory has remained the cornerstone to our current understanding of cosmology and physics. In his theory, Einstein's theory asserts that the creation of singularity points in the curvature of space-time can occur, and these are known as black holes. What is interesting is that Black Holes have recently been theorized to display Marginally trapped surfaces around their event horizon which create a hypersurface that are constructed of quasi-minimal surfaces [2]. In terms of the classical theory, the previous topics of Plateau's problems and mathematical physics were the most progressive advancements.

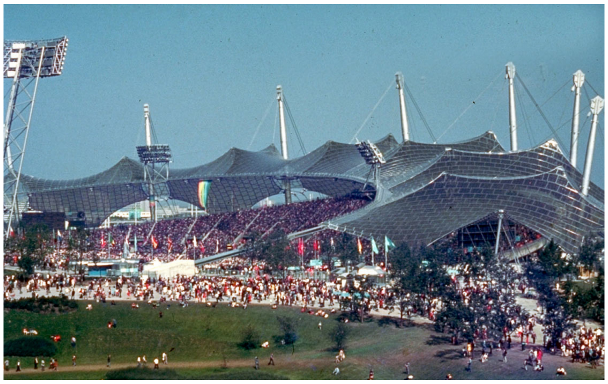
After this *Golden Age*, minimal surface theory began to approach the analysis of surfaces with in higher dimensions and manifolds. Therefore, since the 1980s, Meeks and Perez argue that the field of minimal surfaces is currently in a third age of progression.

5 The 3rd *Golden Age*: 1980-Present

In this current period, the modern theory of minimal surfaces begins to develop from the creation of several branches of mathematics including: Geometric Measure Theory, Conformal Geometry, functional analysis, etc. This period is also emphasized by extensive research on the classical theory of minimal surfaces as well as global problems related to minimal surfaces. The current understanding of minimal surfaces has progressed into discussing areas of minimal submanifolds, minimal surfaces in manifolds of higher dimensions, and the analyzing of constant mean curvature surfaces. Geometric measure theory as well as many other fields of math that developed allowed more surfaces to be explored then previously in *Classical Minimal Surface Theory.* The new fields of mathematics mentioned above now allow for the analysis of surfaces that contain singularities. The types of singularities on surfaces include: self surface intersections, points were the surface has holes, or points that blow up to infinity. Since the details of this third *Golden Age* of the theory is far too extensive to note, it may be more fruitful to discuss how minimal surfaces have led to applications in other research areas and what current problems exist in the theory.

Several articles connect minimal surface theory to certain applications in research fields including: architecture, materials engineering, biology, and physics. Hence, the importance of minimal surface theory as a whole is displayed.

Minimal surfaces have been used in the fields of architecture and materials engineering. Mainly, minimal surfaces with least-area like property globally could reduce the amount of materials needed to build roofs, buildings etc. For instance, Frei Otto used minimal surfaces to construct the 1972 Summer Olympic Games Stadium in Munich Germany. He combined the use of minimal surfaces properties and light weight materials to build magnificent roofing.



Frei Otto also used minimal surfaces in other designs such as the German Pavillon Expo of 1968 and the Kongreshall in Berlin [22].

In addition to architecture, minimal surfaces have also been shown to apply to materials engineering as well as biological systems. According to authors Lu Han and Shunai Che, certain triply periodic minimal surfaces are well connected to the study of natural systems and can be used to create materials from their unique geometries [12]. They connect triply periodic minimal surfaces to recent developments into block copolymer systems and other self assembling systems. Another minimal surface that has been connected to materials science is the gyroid discovered in 1970 by Alan Schoen [13]. The gyroid is an example of one such minimal surface that has been observed in diblock copolymer systems. Here is the visualization of a gyroid:

Lastly, other major fields that minimal surfaces have also been linked to are physics and mathematical theories. Meeks and Perez note several problems that were solved with the aid of minimal surface theory such as the Positive Mass Conjecture, the Penrose Conjecture, Smith Conjecture and the Poincare Conjecture [23]. A more in depth study into these connections is seen in these references ([14], [16], [15]). for more information. All of the aforementioned conjectures have serious consequences in the overall mathematical and physics communities which further emphasize the importance minimal surfaces have played in reality. Thus, the study of minimal surfaces is justified and it is remains to be a lucrative field for knowledge. That is why the study of minimal surfaces should continue into the future.

Lastly, There are various open problems in the theory of minimal surfaces and they would be too numerous to list here. However, several sources go on to note the open problems in the field for research. Refer to these sources for some of the current open problems in the field of minimal surfaces [23], [3], and [7]. Some of these open problems have been solved however the field is still very free for future research. That said, there is sure to be another *Golden Age* for minimal surfaces in the upcoming years. Hence, the study of minimal surfaces is now more than ever a rich field of information in which, hopefully, many mathematicians or those sparked with interest in the subject should indeed attempt to expand the current knowledge of minimal surface theory.

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